Acoustic Field inside a Rigid Sphere with a Point Source

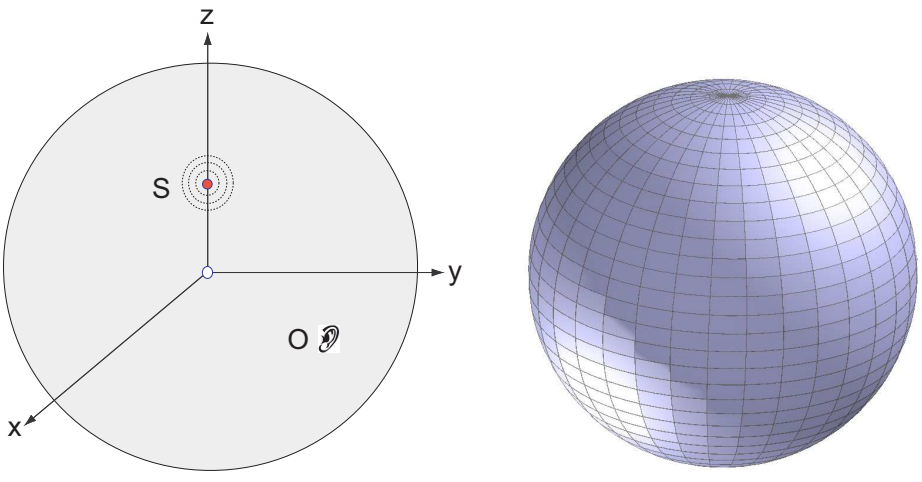
1 Introduction

The main objectives of this Demo Model are to

- Demonstrate the ability of Coustyx to model a rigid sphere with a point source using Coustyx MultiDomain model and solve for the acoustic field distribution inside the sphere.
- Derive analytical solution using modal expansion for a source on the z-axis. For a source at any other point, a simple transformation is applied to make the source lie on the z-axis and obtain the analytical solution.
- Validate Coustyx software by comparing the results from Coustyx to the analytical solutions in the presence of acoustic sources.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium inside the sphere is air with mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$ and sound speed $c = 343 - i*10\,\mathrm{m/s}$. A complex speed of sound introduces damping in the system. The imaginary part of the speed of sound should always be negative for a decaying sound wave. The wavenumber at a frequency ω is given as $k = omega/c$. A monopole source of unit volume velocity is introduced at (0,0,0.35) to simulate the point source in the sphere. The sphere is assumed to be rigid. The BE mesh of the sphere is shown in Figure 1.



(a) Acoustic problem. (b) Boundary element mesh

Figure 1: Rigid sphere with a point source. Note S–source, O–observation point

3 Boundary conditions

In the Coustyx MultiDomain model, the rigid boundary condition is simulated by applying the boundary condition on the sphere as "Uniform Normal Velocity" type with zero amplitude. That is, $v_n = 0$, where v_n is the particle normal velocity on the surface of the sphere in the *Domain Normal* direction. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For this example, the interior domain is the domain of interest; hence, domain normal is pointing away from the sphere interior.

4 Analytical solution

We first compute modes for a rigid sphere (of radius a). These modes are then used in modal expansion to evaluate field point pressure at any point inside the sphere.

4.1 Eigenvalue problem

Table 1: Eigenvalues of a rigid sphere, $k_l(n)$

$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
4.493409	2.081576	3.342094	4.5141	5.646704	6.756456
7.725252	5.94037	7.289932	8.583755	9.840446	11.07021
10.90412	9.20584	10.61386	11.97273	13.29556	14.59055
14.06619	12.40445	13.84611	15.24451	16.60935	17.94718
17.22076	15.57924	17.0429	18.46815	19.86242	21.23107
20.3713	18.74265	20.22186	21.66661	23.0828	24.37266
23.51945	21.8997	23.39049	24.85009	26.28327	27.69372
26.66605	25.05283	26.55259	28.02387	29.47064	30.896
29.8116	28.20336	29.71028	31.19102	32.64889	34.0866
32.95639	31.35209	32.86485	34.35339	35.79049	37.26863

The eigen-function $\Psi_{l,m}(\mathbf{r},n)$ satisfies the Helmholtz equation at any point $\mathbf{r}(\mathbf{r},\theta,\phi)$ inside the sphere

$$\left[\nabla^2 + k_l^2(n)\right]\Psi_{l,m}(\mathbf{r},n) = 0 \tag{1}$$

where $k_l^2(n)$ is the eigenvalue.
The eigen-function should also satisfy the rigid boundary conditions on the surface of the sphere

$$\frac{\partial\Psi_{l,m}(\mathbf{r},n)}{\partial r} = 0, \text{ at r = a} \tag{2}$$

Since the source is assumed to lie on the z-axis, the general form of the eigen-function depends on only $m=0$ modes, that is, there is no dependency with respect to $cos(m\phi)$. The eigen-function is expressed in spherical harmonics as

$$\Psi_l(\mathbf{r},n) = j_l(k_l(n)r)P_l^o(\cos\theta) \tag{3}$$

where j_l is the spherical Bessel function of order l, P_l^o is the associated Legendre function of degree l and order $m=0$.
The eigenvalues $(k_l(n))$ are the roots of the derivative of the spherical Bessel function at $r=a$, that is,

$$j_l'(k_l(n)a) = 0 \tag{4}$$

The above equation is solved numerically using Maple to find the eigenvalues $(k_l(n))$. Table 4.1 shows the first ten eigenvalues for $l=0,...,5$.

4.2 Modal expansion

The acoustic field pressure p at any point $\mathbf{r}(r,\theta,\phi)$ (inside the sphere) due to the presence of a point source at $\mathbf{r}_s(r_s,0,0)$ satisfies the Helmholtz equation

$$\nabla^2 p + k^2 p = \varpi\delta(\mathbf{r}-\mathbf{r}_s) \tag{5}$$

where the source strength $\varpi = ik\rho_o c\beta_o$, β_o is the volume velocity; $\delta(\mathbf{r}-\mathbf{r}_s)$ is the Dirac delta function.
The modal eigen-functions derived above form a complete set. Hence, the acoustic solution p inside the sphere can be approximated as a linear combination of these eigen-functions.

$$\begin{aligned} p &= \sum_n\sum_l A_l\Psi_l(\mathbf{r},n) \\ &= \sum_n\sum_l A_l j_l(k_l r)P_l^o(\cos\theta) \end{aligned} \tag{6}$$

where A_l is the mode participation coefficient.
We compute the mode participation coefficient by substituting Equation 6 into Equation 5, that is,

$$\sum_n \sum_l \left[k^2 - k_l^2(n)\right] A_l j_l(k_l r) P_l^o(\cos\theta) = \varpi\delta(\mathbf{x} - \mathbf{x}_s) \tag{7}$$

We use the following properties to compute A_l. Normalization integral for associated Legendre functions,

$$\int_{-1}^{+1} \left[P_l^m(x)\right]^2 dx = 1$$

Normalization integral for spherical Bessel functions (refer [1]),

$$N_l = \int_0^1 j_l^2(\alpha_l r) r^2 dr = \frac{1}{2\alpha_l^2}\left\{\frac{1}{4} + \alpha_l^2 - \left(l + \frac{1}{2}\right)^2\right\} j_l^2(\alpha_l)$$

To compute A_l, multiply both sides of Equation 7 with $j_l(k_l r) P_l^o(\cos\theta)$ and integrate over the entire volume. Using the properties of various functions described above, along with the properties of Dirac delta function, the coefficient A_l is derived.

$$A_l = \varpi \frac{j_l(k_l r_s) P_l^o(\cos\theta_s)}{\left[k^2 - k_l^2(n)\right] N_l 2\pi} \tag{8}$$

Thus the modal expansion for the pressure at an observation point (r, θ, ϕ) inside the rigid sphere in the presence of a point source on the z-axis$(r_s, 0, 0)$ is,

$$p(r,\theta,\phi) = \sum_n \sum_l \varpi \frac{j_l(k_l r_s) P_l^o(\cos\theta_s)}{\left[k^2 - k_l^2(n)\right] N_l 2\pi} j_l(k_l r) P_l^o(\cos\theta) \tag{9}$$

The pressure in Cartesian coordinates are obtained by applying the following transformations:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}), 0 \le \theta \le \pi \\ \phi &= \arctan(y/x), 0 < \phi \le 2\pi \end{aligned} \tag{10}$$

For a source at an arbitrary point (x_s,y_s,z_s) and the observation point at (x,y,z), we can compute the pressure at the observation point by transforming the axis such that the source lies on (0,0,$\bar{z}_s$). Assume the coordinates of the observation point due to the transformation be ($\bar{x}$,$\bar{y}$,$\bar{z}$). Then,

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = Q \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{11}$$

The transformation matrix Q is given by

$$Q = \begin{bmatrix} \sin\phi_{so} & -\cos\phi_{so} & 0 \\ \cos\theta_{so}\cos\phi_{so} & \cos\theta_{so}\sin\phi_{so} & -\sin\theta_{so} \\ \sin\theta_{so}\cos\phi_{so} & \sin\theta_{so}\sin\phi_{so} & \cos\theta_{so} \end{bmatrix} \tag{12}$$

where θ_{so} and ϕ_{so} are the polar and azimuthal angles of the source before the transformation.

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In the demo model, the analysis is performed for a frequency range of $ka = 1$ to $ka = 10$ with a resolution of $\Delta = 0.2$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results,

along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".
Coustyx MultiDomain model uses Direct BE method to solve the acoustic problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point pressure at an interior observation point (−0.1,−0.3,−0.5) is computed from both Coustyx and analytical methods and are compared in Figure 2 for all frequencies. The comparisons show very good agreement between these two solutions. Figure 2 also shows the resonance peaks at eigen-frequencies of the rigid sphere (Table 4.1).

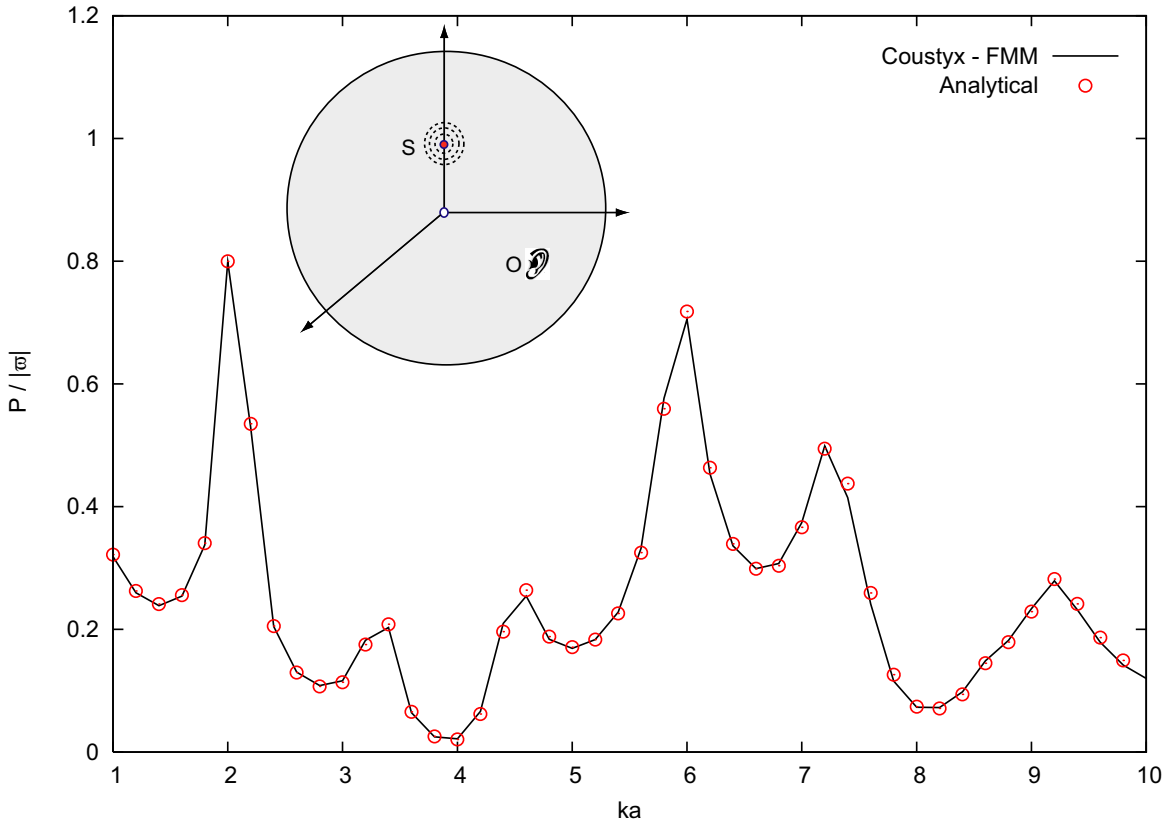


Figure 2: Field pressure comparisons at (−0.1,−0.3,−0.5)inside a rigid sphere with a monopole source at (0,0,0.35) from Coustyx and analytical methods. Note that P is the field point pressure and $\varpi = ik\rho_o c\beta_o$, where β_o is the volume velocity of the monopole source.

References

[1] M. Abramowitz and I. E. Stegun. *Handbook of Mathematical Functions, AMS55.* U.S. Department of Commerce, National Bureau of Standards, 1972.